Appendix.

i. For representation, we'll show that every hyper-revision operation is represented by a 'fall-back' ordering on states along with a particular way of choosing states from a set.

For any s, let $\sigma_{(s)} = s_0, s_1, ...$ be a fall-back ordering—a sequence of states satisfying the following conditions:

(σ_1) $s, \top \in \sigma_s$;
(σ_2) $s \leq s_i$ *if* $s_i \in \sigma_s$;
(σ_3) $s_i \leq s_j$ *if* $s_i, s_j \in \sigma_s$ *and* $i \leq j$.

We define a function $\mathcal{K}$ which, for any σ, s and S, picks out a privileged subset of $S \cup \{\bot\}$ as follows:

$$\mathcal{K}_\sigma(s,S){=}Max(\{s' \in S | s' \leq Min(\{s_i \in \sigma_s | \exists s'' \in S : \bot < s'' \leq s_i\}) \cup \{\bot\})$$

Where $S \nsubseteq \{\bot\}$, $\mathcal{K}_\sigma(s,S)$ is the set containing the weakest states in S which are stronger than the strongest state in σ_s which can be strengthened into a non-absurd state in S. Where $S \subseteq \{\bot\}$, $\mathcal{K}_\sigma(s,S){=}\{\bot\}$.

For any s, let f_s be a choice function. We require that f_s be monotonic—that is, that is, for any X, Y in its domain, if $f_s(X) \in f_s(Y)$ and for all $s' \in Y$ there is some $s'' \in X$ such that $s' \leq s''$, then $f_s(X) = f_s(Y)$

We then prove the following representation theorem.

Theorem 1.

1.1. For any hyper-revision operation $\circledast$ there is some σ and f such that: $s \circledast S = f_s(\mathcal{K}_\sigma(s,S))$, for any s and S.
1.2. For any σ and f there is some hyper-revision operation $\circledast$ such that: $s \circledast S = f_s(\mathcal{K}_\sigma(s,S))$, for any s and S.

Proof of **1.1**: To prove **1.1.**, we start by defining a function σ such that, for any s, $\sigma_s = s_0, s_1 ...$ is the largest sequence satisfying three conditions. First, $s = s_0$. Second, there is no $s_i \in \sigma_s$ such that $s_i = \bot$. Finally, for any $i > 0$:

$$s_i = s \circledast (\mathbb{S} - {\downarrow} s_{i-1})$$

To show that σ_s is indeed a fall-back ordering, we need to show that it satisfies (σ_{1-3}).

We'll start by proving (σ_3) by induction. We need to show that, for any $s_i, s_j \in \sigma_s$, if $i \leq j$, then $s_i \leq s_j$. First, for the base case, observe that for any downward closed set X, $(\mathbb{S} - X)$ is an upward closed set. So $(\mathbb{S} - {\downarrow} s)$ is upward closed.

Furthermore, since $s = s \circledast \uparrow s$ (by ($\circledast_3$)), we know that if $\{s' \in S | s' \leq s\} \nsubseteq \{\bot\}$ then $s \circledast S \leq s$ (by ($\circledast_6$)). Next, suppose for induction that if $\{s' \in S | s' \leq s_i\} \nsubseteq \{\bot\}$, then $s \circledast S \leq s_i$. We'll prove that: (i) if $\{s' \in S | s' \leq s_{i+1}\} \nsubseteq \{\bot\}$, then $s \circledast S \leq s_{i+1}$; and (ii) $s_i \leq s_{i+1}$.

From the fact that $s_{i+1} = s \circledast (\mathbb{S} - \downarrow s_i)$ and $(\mathbb{S} - \downarrow s_i)$ is upward closed, (i) follows immediately by ($\circledast_6$). To prove (ii), consider $\{s_i, s_{i+1}\}$. We know that $s \circledast \{s_i, s_{i+1}\} \leq s_i$, by the inductive hypothesis. And we know that $s_{i+1} \nleq s_i$, since $s_{i+1} \in (\mathbb{S} - \downarrow s_i)$. So $s \circledast \{s_i, s_{i+1}\} = s_i$. But, from (i), we know that $s \circledast \{s_i, s_{i+1}\} \leq s_{i+1}$. So $s_i \leq s_{i+1}$.

(σ_2) follows immediately from (σ_3) along with the fact that $s = s_0$.

Finally, since $s = s_0$, to establish (σ_1) it suffices to demonstrate that $\top \in \sigma_s$. Observe that, by (σ_3), it is guaranteed that $\bigvee\{\downarrow s_k | k \leq i\} = \downarrow s_i$. So, for any sub-sequence $Z \subseteq \sigma_s$, we know that $s \circledast (\mathbb{S} - \bigvee\{\downarrow s_i | s_i \in Z\}) \in \sigma_s$ iff $\bigvee\{\downarrow s_i | s_i \in Z\} \neq \mathbb{S}$. By the construction of σ, $\{\downarrow s_i | s_i < \top\}$ is a sub-sequence of σ_s. Now consider $s \circledast (\mathbb{S} - \bigvee\{\downarrow s_i | s_i < \top\})$—call this state $\hat{s}$. Since $\bigvee\{\downarrow s_i | s_i < \top\} \neq \mathbb{S}$, it follows that $\hat{s} \in \sigma_s$. By ($\circledast_3$), we can know that $\hat{s} \nless \top$ (since $\hat{s} \notin \bigvee\{\downarrow s_i | s_i < \top\}$). So $\hat{s} = \top$.[1]

Next, we'll show that $s \circledast S$ is a member of $\mathcal{K}_\sigma(s, S)$. That is, $s \circledast S$ is amongst the weakest of (i) the states in S which are stronger than the strongest $s_i \in \sigma_s$ which can be strengthened into a non-absurd state in S and (ii) $\bot$. First, consider the case where $S \subseteq \{\bot\}$. Then we know that there is no $s_i \in \sigma_s$ which can be strengthed into a non-absurd element of S. So $\mathcal{K}_\sigma(s, S) = \{\bot\}$. But, by ($\circledast_1$), $s \circledast S = \bot$. So $s \circledast S \in \mathcal{K}_\sigma(s, S)$.

Next, consider the case where $S \nsubseteq \{\bot\}$. Then we know that there is some $s_i \in \sigma_s$ which can be strengthened into a non-absurd element of S. Let s_k be the strongest such element of σ_s. We know, by the construction of σ, that s_k is the result of hyper-revising s with some upward closed set. So, by ($\circledast_6$), it follows that for any S' containing a non-absurd state stronger than s_k, $s \circledast S' \leq s_k$. But S contains a non-absurd state stronger than s_k. So $s \circledast S \leq s_k$.

Now consider $\{s' \in S | \bot < s' \leq s_k\}$. We know that this state is a subset of $(\mathbb{S} - \downarrow s_{k-1})$, since otherwise s_k would not be minimal. So we know, by ($\circledast_5$), that there is no $s'' \in S$ such that $s \circledast \{s' \in S | \bot < s' \leq s_k\} < s'' \leq s \circledast (\mathbb{S} - \downarrow s_{k-1}) = s_k$. That is, $s \circledast \{s' \in S | \bot < s' \leq s_k\}$ is amongst the weakest states in S which are stronger than s_k. But recall that $s \circledast S \in \{s' \in S | \bot < s' \leq s_k\}$. So, it follows, by ($\circledast_4$), that $s \circledast \{s' \in S | \bot < s' \leq s_k\} = s \circledast S$. So $s \circledast S \in Max\{s' \in S | \bot < s' \leq s_k\} = \mathcal{K}_\sigma(s, S)$.

Finally, we need to show that, for any s, it is possible to construct a monotonic choice function f_s which, given any S, selects $s \circledast S$ from amongst $\mathcal{K}_\sigma(s, S)$. We will start by showing that, for any S' such that $\mathcal{K}_\sigma(s, S') \subseteq \mathcal{K}_\sigma(s, S)$ and $s \circledast S \in \mathcal{K}_\sigma(s, S')$, $s \circledast S' = s \circledast S$. The case where $S \subseteq \{\bot\}$ is trivial. So suppose

[1] Observe that since $s \circledast (\mathbb{S} - \downarrow \top) = \bot$, it follows that there is no $s_j \in \sigma_s$ such that $s_i = \top$ and $i < j$.

that $S \not\subseteq \{\bot\}$. By ($\circledast_2$) and the assumption that $s \circledast S \in \mathcal{K}_\sigma(s, S')$, we also have that $S' \not\subseteq \{\bot\}$. By the above reasoning, we know that $s \circledast S' \in \mathcal{K}_\sigma(s, S') \subseteq S'$ (and *mutatis mutandis*, for S). So ($\circledast_4$) ensures that $s \circledast \mathcal{K}_\sigma(s, S) = s \circledast S$ and $s \circledast \mathcal{K}_\sigma(s, S') = s \circledast S'$. We also know by assumption that $\mathcal{K}_\sigma(s, S') \subseteq \mathcal{K}_\sigma(s, S)$ and $s \circledast \mathcal{K}_\sigma(s, S) \in \mathcal{K}_\sigma(s, S')$. So ($\circledast_4$) also ensures that $s \circledast \mathcal{K}_\sigma(s, S') = s \circledast \mathcal{K}_\sigma(s, S)$. Combining the two, we can conclude that $s \circledast S = s \circledast S'$. So, for any s, we can construct a function f_s with the domain $\{\mathcal{K}_\sigma(s, S) | S \in \mathcal{P}(\mathbb{S})\}$ such that, for any S, $f_s(\mathcal{K}_\sigma(s, S)) = s \circledast S$. Trivially, f_s is a choice function. We need to show that f_s is monotonic. Suppose that $f_s(\mathcal{K}_\sigma(s, S)) \in \mathcal{K}_\sigma(s, S')$ and for all $s' \in S'$ there is some $s'' \in S$ such that $s' \leq s''$. We know that the minimal $s_k \in \sigma_s$ which can be strengthened into a non-absurd element is the same for S and S'. So it is then easy to see that $\mathcal{K}_\sigma(s, S) = \mathcal{K}_\sigma(s, S \cup S')$. By the above reasoning, it follows that $s \circledast S = s \circledast (S \cup S')$. But observe that $S' \subseteq S \cup S'$ and, by hypothesis, $s \circledast (S \cup S') \in S'$. So by ($\circledast_4$), it follows that $s \circledast S = s \circledast S'$. So f_s is monotonic.

So, for an arbitrary hyper-revision operation $\circledast$, it is possible, for each s, to construct a fall-back ordering σ_s and monotonic choice function f_s such that $s \circledast S = f_s(\mathcal{K}_\sigma(s, S))$. We let σ and f be the functions mapping each s to those objects.

Proof of **1.2**: Let $\odot$ be an operation such that, for some arbitrary σ and f, $s \odot S = f_s(\mathcal{K}_\sigma(s, S'))$. To prove **1.2**, we need to show that $\odot$ is a hyper-revision operation.

As can be easily checked, $\mathcal{K}_\sigma(s, S) \subseteq S$, if $S \neq$ and $\mathcal{K}_\sigma(s, S) = \{\bot\}$, otherwise. So $\odot$ satisfies ($\circledast_1$).

Next, observe that if $S \not\subseteq \{\bot\}$ then there is some non-absurd $s' \in S$ such that s' is weaker than the strongest $s_i \in \sigma_s$ which can be strengthened into a non-absurd element of S. Since $\bot < s'$, $\bot \notin \mathcal{K}_\sigma(s, S)$. So $\odot$ satisfies ($\circledast_2$).

Suppose that $s \in S$. Then the strongest state in σ_s which can be strengthened into a non-absurd state in S is $s_0 = s$. Furthermore, since s is the weakest state in S which is at least as strong as s, $\mathcal{K}_\sigma(s, S) = \{s\}$. So $\odot$ satisfies ($\circledast_3$).

Suppose that $s \odot S \in S \cap S'$. Then we know that $f_s(\mathcal{K}_\sigma(s, S)) \in \mathcal{K}_\sigma(s, S \cap S')$. Observe that the minimal states in σ_s which can be strengthened into a non-absurd member is the same for S and $S \cap S'$. So it is easy to see that for any $s' \in \mathcal{K}_\sigma(s, S \cap S')$, there must be some $s'' \in \mathcal{K}_\sigma(s, S)$ such that $s' \leq s''$. So, since f_s is monotonic, it follows that $f_s(\mathcal{K}_\sigma(s, S \cap S')) = f_s(\mathcal{K}_\sigma(s, S))$. So $\odot$ satisfies ($\circledast_4$).

For reductio, suppose that there is some $s' \in S \cap S'$ such that $s \odot S \cap S' < s' \leq s \odot S$. Since $s \odot S \cap S' \leq s \odot S$, we know that the minimal $s_i \in \sigma_s$ which can be strengthened into a non-absurd state in the set must be the same for $S \cap S'$ and S. Call that state s_k. We know that $s \odot S \leq s_k$. So $s \odot S \cap S' < s' \leq s_k$. But then $s \odot S \cap S'$ is not maximal, so $s \odot S \cap S' \notin \mathcal{K}_\sigma(s, S \cap S')$. But by the construction of $\odot$, $s \odot S \cap S' = f_s(\mathcal{K}_\sigma(s, S \cap S'))$ and f_s is a choice function.

Contradiction. So there is no such $s' \in S \cap S'$. So $\odot$ satisfies ($\circledast_5$).

Finally, we need to show that $s \odot S' \leq s \odot \uparrow S$, wherever $\{s' \in S' | s' \leq s \odot \uparrow S\} \not\subseteq \{\bot\}$. Consider the minimal $s_i \in \sigma_s$ which can be strengthened into some non-absurd element of $\uparrow S$. Since $\uparrow S$ is upward closed, we know that $\mathcal{K}_\sigma(s, \uparrow S) = \{s_i\}$. So it follows that $s \odot \uparrow S = s_i$. Now take an arbitrary S' such that $\{s' \in S' | s' \leq s_i\} \not\subseteq \{\bot\}$. Then we know that, for the minimal s_j which can be strengthened into a non-absurd element of S', $s_j \leq s_i$. And we know that for any $s'' \in \mathcal{K}_\sigma(s, S')$, $s' \leq s_j \leq s \odot \uparrow S$. So it follows that $\odot$ satisfies ($\circledast_6$).

Since $\odot$ satisfies ($\circledast_1$)-($\circledast_6$), $\odot$ is a hyper-revision operation. □

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ii. Our next task is to prove **Fact 3**:

> **Fact 3.** For any hyper-revision operation, $\circledast$, there is some AGM revision operation, $*$, such that for any $s, s' \in \mathbb{S}$: $s \circledast (\downarrow s') = s * s'$.

Proof: Let $\odot : (\mathbb{S} \times \mathbb{S}) \to \mathbb{S}$ be an operation such that $s \odot s' = s \circledast \downarrow (s')$. We prove need to prove that $\odot$ is an AGM revision operation. That is, we must show that $\odot$ satisfies ($*_{1-4}$)

First note that, for any $s, s' \in \mathbb{S}$, we know that $s \odot s' \in \downarrow(s')$ (by ($\circledast_1$)). So, it follows that $\odot$ satisfies ($*_1$). We also know that $s \circledast \downarrow s' = \bot$ only if $s' = \bot$ (by ($\circledast_2$)). So it follows that $\odot$ satisfies ($*_2$), too.

Observe that $\{s'' \in \downarrow s' | s'' \leq s\} \not\subseteq \{\bot\}$ iff $s \wedge s' \neq \bot$. Furthermore, we know that if $\{s'' \in \downarrow s' | s'' \leq s\} \not\subseteq$, then $s \circledast \downarrow s' = Max\{s'' \in \downarrow s' | s'' \leq s\} = s \wedge s'$ (by ($\circledast_3$), ($\circledast_5$) and ($\circledast_6$)). So it follows that $\odot$ satisfies ($*_3$), too.

Finally, suppose that $(s \odot s') \wedge s'' \neq \bot$. We need to show that $s \odot (s' \wedge s'') = (s \odot s') \wedge s''$. By hypothesis, and the fact that $\odot$ satisfies, ($*_1$), we can be sure that $s' \neq \bot$. So we know that $s \circledast \downarrow s' = s \circledast (\downarrow s' - \{\bot\})$ (by ($\circledast_2$) and ($\circledast_4$)). Observe that $(\downarrow s' - \{\bot\}) \subseteq \uparrow(\downarrow s' - \{\bot\})$ and let $s_i = s \circledast \uparrow(\downarrow(s') - \{\bot\})$. From ($\circledast_6$), it follows that $s \circledast \downarrow s' \leq s_i$. Furthermore, by ($\circledast_5$), $s \odot s'$ is amongst the weakest elements of $\downarrow s'$ which are stronger than s_i. But $s' \wedge s_i$ is the unique weakest such state. So $s \odot s' = s_i \wedge s'$.

Next, observe that if $(s \odot s') \wedge s'' \neq \bot$, then $\downarrow(s' \wedge s'')) \cap \downarrow s_i \not\subseteq \{\bot\}$. So $\downarrow(s' \wedge s'')) \leq s_i$, by ($\circledast_6$). By hypothesis, $s' \wedge s'' \neq \bot$. So we know that $s \circledast \downarrow(s' \wedge s'') = s \circledast (\downarrow(s' \wedge s'') - \{\bot\})$ (by ($\circledast_2$) and ($\circledast_4$)). Furthermore, $(\downarrow(s' \wedge s'') - \{\bot\}) \subseteq \uparrow(\downarrow s' - \{\bot\})$. So $s \odot (s' \wedge s'')$ is amongst the weakest elements of $\downarrow(s' \wedge s'')$ which are stronger than s_i. But $s_i \wedge (s' \wedge s'')$ is the unique weakest such state. So $s \odot (s' \wedge s'') = s_i \wedge (s' \wedge s'')$. But $\wedge$ is associative: that is, $s_i \wedge (s' \wedge s'') = (s_i \wedge s') \wedge s''$. So $(s \odot s') \wedge s'' = s \odot (s' \wedge s'')$. So it follows that $\odot$ satisfies ($*_4$).

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iii. Finally, we show that hyper-revision of a state can be decomposed into sequential operations of hyper-subtraction and hyper-addition:

> **Fact 4.** For any information state, s, and hyper-revision operation, $\circledast$, there is some hyper-contraction operation, $\ominus$, and hyper-expansion operation, $\oplus$, such that for any S: $\quad s \circledast S = (s \ominus \overline{S}) \oplus S$

To prove **Fact 4**, we introduce, for each hyper-revision operation, a corrresponding operation, $\ominus$, defined such that:

$$s \ominus S = s \circledast \uparrow\!(s \vee (s \circledast \overline{S}))$$

And a set of operations, Δ, defined such that:

$$\Delta = \{\oplus_i \colon s \oplus_i S \in Max(\downarrow\! s \cap S \cup \{\bot\})\}$$

It is easy to observe that $\ominus$ is a hyper-contraction operation (that is, $s \leq s \ominus S$) and that, for each $\oplus_i \in \Delta$, $\oplus_i$ is a hyper-expansion operation (that is, $s \oplus_i S \leq s$).

For a proof of **Fact 4**, we need to demonstrate that there is some $\oplus_i \in \Delta$ such that, for any S, $s \circledast S = (s \ominus \overline{S}) \oplus_i S$. To demonstrate this it suffices, in turn, to show that $s \circledast S \in Max(\downarrow\!(s \ominus \overline{S}) \cap S)$. That is, we simply need to prove that $s \circledast S$ is in the weakest states in S which are at least as strong as $s \circledast \uparrow\!(s \vee (s \circledast S))$. In case $S \subseteq \{\bot\}$, the proof is trivial. So suppose otherwise.

Consider the weakest s_i such that $s \leq s_i < (s \ominus \overline{S})$ and $(s \circledast \uparrow\! s_i) = s_i$. We are first going to show that there is no non-absurd state in S which is at least as strong as s_i. Suppose otherwise, for reductio. Then $s \circledast S \leq s_i$, by ($\circledast_6$). So it follows that $s \vee s \circledast S \leq s_i$. Yet in that case, $s \circledast \uparrow\!(s \vee (s \circledast S)) = s_i$, by ($\circledast_4$). Afterall, we know that $s \circledast \uparrow\!(s \vee (s \circledast S) \in \uparrow\! s_i$ (since $s_i \leq (s \circledast \uparrow\!(s \vee (s \circledast S)))$. And we also know that $\uparrow s_i \subseteq \uparrow\!(s \vee (s \circledast S))$. Yet, by hypothesis, $s \circledast \uparrow\!(s \vee (s \circledast S)) \neq s_i$. Contradiction. So $\downarrow\!(s_i) \cap S \subseteq \{\bot\}$.

Next, consider $s \circledast \overline{\downarrow\! s_i}$. And we know, from ($\circledast_6$), that $s \circledast \overline{\downarrow\! s_i} \leq s \ominus \overline{S}$. So either $s_i < s \circledast \overline{\downarrow\! s_i} < s \ominus \overline{S}$ or $s \circledast \overline{\downarrow\! s_i} = s \ominus \overline{S}$. We know, for any S, that $s \circledast \uparrow\!(s \circledast \uparrow\! S) = s \circledast \uparrow\! S$ (by ($\circledast_4$)). Since $\overline{\downarrow\! s_i}$ is an upward-closed set, it follows that $s \circledast \uparrow\!(s \circledast \overline{\downarrow\! s_i}) = s \circledast \overline{\downarrow\! s_i}$. But there is no s_j such that $s_i < s_j < s \ominus \overline{S}$ and $s \circledast \uparrow\! s_j = s_j$. So $s \circledast \overline{\downarrow\! s_i} = s \ominus \overline{S}$.

But now observe that, since $\downarrow\!(s_i) \cap S \subseteq \{\bot\}$, we know that $\{s' \in S : \bot < s' \leq s \ominus \overline{S}\} \subseteq \overline{\downarrow\! s_i}$. So, by ($\circledast_5$), $s \circledast \{s' \in S : \bot < s' \leq s \ominus \overline{S}\}$ is among the weakest elements of S stronger than $s \circledast \downarrow\! s_i = s \ominus \overline{S}$. Yet since we know that $s \circledast S \in \{s' \in S : \bot < s' \leq s \ominus \overline{S}\}$, it follows from ($\circledast_4$) that $s \circledast S = s \circledast \{s' \in S : \bot < s' \leq s \ominus \overline{S}\}$. So $s \circledast S$ is among the weakest states in S which are stronger than $s \ominus \overline{S}$. $\square$